TECHNICAL NOTE

Thermal response of a periodic boundary layer near an axisymmetric stagnation point on a circular cylinder

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Boundary layer solutions are provided to study the time-mean heat transfer characteristics in a laminar flow in the vicinity of an axisymmetric stagnation point. The velocity of the oncoming flow is assumed to oscillate relative to the body. Different solutions are constructed for the small and high values of the reduced frequency parameter. Numerical solutions for the temperature functions are presented, and the wall values of the thermal gradients are tabulated.

Keywords: periodic boundary layer; stagnation flows; convective heat transfer

Introduction

One of the most significant studies of laminar boundary layers under the influence of a purely time-dependent free-stream oscillation was reported by Lighthill.[1] His analysis employed a linearization for small oscillation amplitudes. Lin[2] considered the effect of finite amplitude oscillation on a flow field. Mori and Tokuda[3] investigated the heat transfer from an oscillating cylinder. Recently, Gorla *et al.*[4] examined the fluid flow characteristics in an oscillating laminar boundary layer in the vicinity of an axisymmetric stagnation point by means of a boundary layer approximation. They evaluated the amplitude and phase angle of the wall skin friction fluctuation for a wide range of the reduced frequency of oscillation.

The present work deals with the time-mean heat transfer characteristics of the periodic boundary layer near an axisymmetric stagnation point on a circular cylinder. The analysis considers the case when the fluctuations in the external flow are produced by those of the oncoming stream. Figure 1 shows a cylinder described by $r=a$ in cylindrical polar coordinates. The flow is axisymmetric about the z axis and also symmetric about the $z=0$ plane. The stagnation line is at $z=0$, $r=a$. Different solutions are obtained for the small and high values of the reduced frequency parameter. The range of Reynolds numbers considered was from 0.01 to 100 for a Prandtl number of 0.7. Numerical solutions for the temperature functions and the Nusselt number are presented.

The flow configuration described in this paper is applied in certain cooling and quenching processes.

Governing equations

Consider a laminar, incompressible, unsteady flow at an axisymmetric stagnation point on a circular cylinder. It is assumed that the properties of the fluid are constant, and viscous dissipation may be neglected. Figure 1 shows the flow model and the coordinate system. The governing equations under the boundary layer approximation are given by

Mass

$$r\frac{\partial w}{\partial z}+\frac{\partial (ru)}{\partial r}=0 \tag{1}$$

Momentum

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right] \tag{2}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right] \tag{3}$$

Energy

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}=\alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] \tag{4}$$

The dimensionless temperature is defined as $\theta=(T-T_\infty)/(T_w-T_\infty)$. The boundary conditions are given by

1. $r=a$: $\quad u=w=0,\ T=T_w$.
2. $r\to\infty$: $\quad u=U_e=-A(r-a^2/r)(1+e^{i\Omega t})$

$$w=W_e=2Az(1+\varepsilon e^{i\Omega t})$$

$$\mathrm{T}=\mathrm{T}_\infty \tag{5}$$

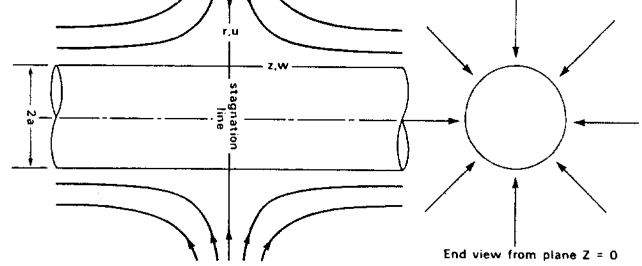

Figure 1 Coordinate system and flow development

When ε is small compared with unity, u, w, and θ may be expanded as

$$u(r,z,t)=u_0(r,z)+\varepsilon u_1(r,z,t)+\varepsilon^2 u_2(r,z,t)+\cdots$$
$$w(r,z,t)=w_0(r,z)+\varepsilon w_1(r,z,t)+\varepsilon^2 w_2(r,z,t)+\cdots$$
$$\theta(r,z,t)=\theta_0(r,z)+\varepsilon\theta_1(r,z,t)+\varepsilon^2\theta_2(r,z,t)+\cdots \quad (6)$$

Substituting Equation 6 into Equations 1, 2, and 3 and equating terms of the same order of ε, we obtain sets of differential equations. Since the equations governing the velocity field are given by Gorla *et al.*,[4] we will not repeat them here. For the heat transfer problem, the zeroth-order equation is

$$u_0\frac{\partial\theta_0}{\partial r}+w_0\frac{\partial\theta_0}{\partial z}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta_0}{\partial r}\right) \quad (7)$$

The boundary conditions for the zeroth-order equation may be written as

$$r=a: \quad \theta_0=1$$
$$r\rightarrow\infty: \quad \theta_0\rightarrow 0 \quad (8)$$

The first-order heat transfer equation is

$$\frac{\partial\theta_1}{\partial t}+u_0\frac{\partial\theta_1}{\partial r}+u_1\frac{\partial\theta_0}{\partial r}+w_0\frac{\partial\theta_1}{\partial z}+w_1\frac{\partial\theta_0}{\partial z}=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta_1}{\partial r}\right) \quad (9)$$

The boundary conditions for the first-order equation are

$$r=a: \quad \theta_1=0$$
$$r\rightarrow\infty: \quad \theta_1\rightarrow 0 \quad (10)$$

Coordinate transformation and solution

We now define

$$\eta=\left(\frac{r}{a}\right)^2$$
$$u_0=-Aa\eta^{-1/2}f(\eta)$$
$$w_0=2Af'(\eta)z$$
$$u_1=-Aa\eta^{-1/2}g(\eta)e^{i\Omega t} \quad (11)$$
$$w_1=2Ag'(\eta)e^{i\Omega t}$$
$$\theta_1=\psi(\eta)e^{i\Omega t}$$
$$\sigma=\frac{\Omega}{A}$$
$$\mathrm{Re}=\frac{Aa^2}{2\nu}$$

Substituting the expressions in Equation 11 into the zeroth-order equations, we have

$$\eta f'''+f''+\mathrm{Re}[1+ff''-(f')^2]=0 \quad (12)$$
$$\eta\theta_0''+[1+(\mathrm{Re}\cdot\mathrm{Pr})f]\theta_0'=0 \quad (13)$$

The primes denote differentiation with respect to η only. The transformed boundary conditions are

$$f(1)=f'(1)=0,\quad \theta_0(0)=1 \quad\text{and}\quad f'(\infty)=1, \quad\text{and}\quad \theta_0(\infty)=0 \quad (14)$$

The numerical solution for Equations 12 and 13 is well known (see Refs. 5, 6 and 7), so these details will not be repeated.

After substituting the expressions in Equation 11 into the first-order Equation 9, we have

$$\eta\psi''+[1+(\mathrm{Re}\cdot\mathrm{Pr})f]\psi'+(\mathrm{Re}\cdot\mathrm{Pr})g\theta_0'-\frac{i\sigma(\mathrm{Re}\cdot\mathrm{Pr})}{2}\psi=0 \quad (15)$$

with transformed boundary conditions being given by

$$\psi(1)=0,\quad \psi(\infty)=0 \quad (16)$$

Since Equation 15 contains the frequency parameter σ, solutions are presented for small- as well as large-frequency cases.

Small-frequency case

When $\sigma\ll 1$, we assume that

$$\psi(\eta)=\psi_0(\eta)+(i\sigma)\psi_1(\eta)+(i\sigma)^2\psi_2(\eta)+\cdots \quad (17)$$

From Equations 17 and 15 we have for the thermal problem:

$$\eta\psi_0''+[1+(\mathrm{Re}\cdot\mathrm{Pr})f]\psi_0'+(\mathrm{Re}\cdot\mathrm{Pr})g_0\theta_0'=0 \quad (18)$$
$$\eta\psi_1''+[1+(\mathrm{Re}\cdot\mathrm{Pr})f]\psi_1'+(\mathrm{Re}\cdot\mathrm{Pr})g_1\theta_1'-\left(\frac{\mathrm{Re}\cdot\mathrm{Pr}}{2}\right)\psi_0=0 \quad (19)$$
$$\eta\psi_2''+[1+(\mathrm{Re}\cdot\mathrm{Pr})f]\psi_2'+(\mathrm{Re}\cdot\mathrm{Pr})g_2\theta_2'-\left(\frac{\mathrm{Re}\cdot\mathrm{Pr}}{2}\right)\psi_1=0 \quad (20)$$

Notation

A	Constant used in Equation 4
a	Radius of cylinder
C	Amplitude of the fluctuating skin friction
C_h	Amplitude of heat transfer fluctuations
f, g	Velocity profile functions
P	Pressure
Re	Reynolds number
r	Coordinate normal to the cylindrical surface
T	Temperature
t	Time
u	Velocity component in r direction
w	Velocity component in z direction
z	Coordinate parallel to the wall
η	Dimensionless coordinate
θ, ψ	Dimensionless temperature
ϕ	Phase angle of heat transfer fluctuations
ϕ_h	Phase angle of heat transfer fluctuations
μ	Dynamic viscosity
ν	Kinematic viscosity
ρ	Fluid density
Ω	Frequency of oscillation
σ	Reduced frequency parameter
ε	Amplitude of oscillating velocity

Subscripts

w	Conditions at the wall
∞	Conditions far away from the wall

Table 1 Values of $Re^{-1/2}\theta'_0(1)$ and $Re^{-1/2}\psi'_n(1)$ for Pr=0.7 and various values of Re

Re	$Re^{-1/2}\theta'_0(1)$	$Re^{-1/2}\psi'_0(1)$	$Re^{-1/2}\psi'_0(1)$	$Re^{-1/2}\psi'_2(1)$	$Re^{-1/2}\psi'_3(1)$	$Re^{-1/2}\psi'_4(1)$
0.01	−2.105002	−0.259662	0.0485484	−0.00863075	41.308546	−32.803059
0.1	−1.105455	−0.212122	0.0459780	−0.00939101	7.523121	−3.00634381
1.0	−0.715880	−0.194671	0.0462737	−0.0101652	1.071604	−0.280743
10.0	−0.570303	−0.193794	0.0480040	−0.01707189	0.124018	−0.0259809
100.0	−0.520036	−0.184742	0.0439125	−0.00958815	0.0147153	−0.00290194

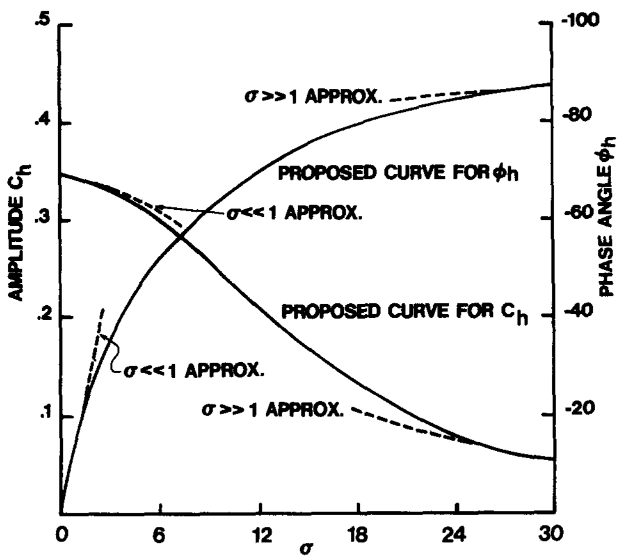

Figure 2 Amplitude C_h and phase angle ϕ_h of the fluctuating component of heat transfer for Pr=0.7 and Re=100 (--- Lighthill's results)

$$\eta\psi_3'' + [1+(\text{Re}\cdot\text{Pr})f]\psi_3' + (\text{Re}\cdot\text{Pr})g_3\theta_3' - \left(\frac{\text{Re}\cdot\text{Pr}}{2}\right)\psi_2 = 0 \tag{21}$$

$$\eta\psi_4'' + [1+(\text{Re}\cdot\text{Pr})f]\psi_4' + (\text{Re}\cdot\text{Pr})g_4\theta_4' - \left(\frac{\text{Re}\cdot\text{Pr}}{2}\right)\psi_3 = 0 \tag{22}$$

etc.

The boundary conditions are

$$\psi_i(1)=0, \quad \psi_i(\infty)=0 \quad \text{for } i\geqslant 0 \tag{23}$$

Equations 18–22 are solved by means of the fourth-order Runge-Kutta numerical procedure on an IBM 370 computer. Re and Pr were treated as prescribable parameters.

The numerical results for the thermal problem are obtained for Pr=0.7 while Re varied from 0.01 to 100. The values $\theta'_0(1)$, $\psi_0(1)$, $\psi'_1(1)$, $\psi'_2(1)$, $\psi'_3(1)$, $\psi'_4(1)$ are tabulated in Table 1. The local Nusselt number for this case may be written as

$$\begin{aligned}\text{Nu}_z\cdot\text{Re}_z^{-1/2} &= -2\,\text{Re}^{1/2}\{\theta'_0(1)+\varepsilon^{i\Omega t}\psi'(1)+\cdots\}\\ &= -2\text{Re}^{1/2}[\theta'_0(1)+\varepsilon e^{i\Omega t}\{\psi'_0(1)+(i\sigma)\psi'_1(1)\\ &\quad +(i\sigma)^2\psi'_2(1)+(i\sigma)^3\psi'_3(1)\\ &\quad +(i\sigma)^4\psi'_4(1)+\cdots\}]\end{aligned} \tag{24}$$

For the low-frequency heat transfer fluctuation we may write

$$\frac{\psi'(1)}{\theta'_0(1)} = \frac{\psi'_0(1)+(i\sigma)\psi'_1(1)+(i\sigma)^2\psi'_2(1)+(i\sigma)^3\psi'_3(1)+\cdots}{\theta'_0(1)} \tag{25}$$

High-frequency case

For this case it may be shown that the large-frequency solution for ψ is given by

$$\psi = \frac{1}{i\sigma}\left\{\left[2(\eta-1)-\left(\frac{1}{2\text{Re}}\right)^{1/2}\cdot\frac{1}{\sqrt{i\sigma}}+\cdots\right] + \frac{\text{Pr}\,f'e^R}{\sqrt{i\sigma}} + \left(\frac{1}{2\text{Re}}\right)^{1/2}\frac{1}{\sqrt{i\sigma}}e^{S(\eta)}\right\} \tag{26}$$

where

$$S(\eta) = -\sqrt{2\text{Re}}\,\sqrt{i\sigma\text{Pr}}(\eta^{1/2}-1) + \frac{\text{Pr}\cdot\text{Re}}{2}\int_1^\eta \frac{f}{\eta}\,d\eta$$

and

$$R = -\sqrt{2\text{Re}}\,\sqrt{i\sigma}(\eta^{1/2}-1) - \left[\frac{3}{4}\left(\frac{1}{\eta}-1\right)+\frac{3}{4}\ln\eta+\frac{\text{Re}}{2}\int_1^\eta\frac{f}{\eta}\,d\eta\right]$$

The amplitude C_h and phase angle ϕ_h of heat transfer fluctuations of order ε are shown in Figure 2 for Re=100 and Pr=0.7.

Terms $O(\varepsilon^2)$ will be of interest, and steady streaming as in acoustics may be obtained.* This will be the subject matter of a future investigation. It is possible also that Equation 15 may have eigensolutions for certain values of (σRe). In that case the expansions given by Equation 6 would be subcritical, and a new expansion should be developed for the critical region, which may yield interesting stability analysis. These aspects will be considered in a future investigation.

Results and concluding remarks

In this paper, a study has been made of the response of the temperature field in the laminar boundary layer near an axisymmetric stagnation point on a circular cylinder as a result of the mainstream oscillation. Solutions are obtained for small as well as large frequencies under the assumption of small amplitude oscillation. Numerical solutions are presented for the temperature field and Nusselt number for a wide range of the Reynolds and Prandtl numbers.

Figure 2 shows the amplitude and phase angle of heat transfer fluctuations of order ε. The results obtained in this paper indicate that the amplitude and phase advance of Nusselt number fluctuation decrease and increase, respectively, with the frequency of oscillation of the mainstream, and an asymptotic phase angle of 90° is attained at very large frequency for Re from 0.01 to 100. Figure 2 displays this result for Re=100.

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